On Lebesgue measure of integral self-affine sets

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January 19, 2011

Abstract

Let A be an expanding integer $n \times n$ matrix and D be a finite subset of $\mathbb{Z}^n$. The self-affine set $T = T(A, D)$ is the unique compact set satisfying the equality $A(T) = \cup_{d\in D}(T + d)$. We present an effective algorithm to compute the Lebesgue measure of the self-affine set T, the measure of the intersection $T \cap (T + u)$ for $u \in \mathbb{Z}^n$, and the measure of the intersection of self-affine sets $T(A, D_1) \cap T(A, D_2)$ for different sets $D_1, D_2 \subset \mathbb{Z}^n$.

Keywords: self-affine set, tile, graph-directed system, self-similar action.
Mathematics Subject Classification 2000: 28A80, 52C22

Let A be an expanding integer $n \times n$ matrix, where expanding means that every eigenvalue has modulus greater than 1, and let D be a finite subset of $\mathbb{Z}^n$. There exists a unique nonempty compact set $T = T(A, D) \subset \mathbb{R}^n$, called (integral) *self-affine set*, satisfying $A(T) = \cup_{d\in D}(T + d)$. It can be given explicitly by

$$T = \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : d_k \in D \right\}.$$

The self-affine set T with $|D| = |\det A|$ and of positive Lebesgue measure is called a *self-affine tile*. Self-affine tiles were intensively studied for the last two decades in the context of self-replicating tilings, radix systems, Haar-type wavelets, etc.

The question of how to find the Lebesgue measure $\lambda(T)$ of the self-affine set T was considered by Lagarias and Wang in [7], where some partial cases were studied. In particular, it was shown that self-affine tiles have integer Lebesgue measure. He, Lau and Rao [4] reduced the problem of finding $\lambda(T)$ to the case when D is a coset transversal for $\mathbb{Z}^n/A(\mathbb{Z}^n)$. The last case was treated by Gabardo and Yu [3] and in more general settings

*The author was partially supported by NSF grant 0503688

by Bondarenko and Kravchenko [1]. The positivity of the Lebesgue measure of self-affine sets was also studied in [8, 6, 2].

In this note, we present a simple method to compute the Lebesgue measure $\lambda(T)$ of the self-affine set T. We construct a finite labeled graph (automaton) and show that $\lambda(T)$ is equal to the uniform Bernoulli measure of the left-infinite sequences which can be read along paths in this graph. Similar graphs when D is a coset transversal were constructed in [3, 10] and other papers. In addition this method allows to find the measure of the intersection $T \cap (T+u)$ for $u \in \mathbb{Z}^n$, and the measure of the intersection of self-affine sets $T(A, D_1) \cap T(A, D_2)$ for different sets $D_1, D_2 \subset \mathbb{Z}^n$. Our construction seems to be very natural and actually works for any contracting self-similar group action (here the self-affine sets correspond to the self-similar actions of $\mathbb{Z}^n$, see [9, Section 6.2] and [1]).

We proceed as follows. If the set D does not contain all coset representatives of $\mathbb{Z}^n/A(\mathbb{Z}^n)$, we extend it to the set $K \supset D$ which does, and choose a coset transversal $C \subset K$.

Construct a directed labeled graph (automaton) $\Gamma = \Gamma(A, K)$ with the set of vertices $\mathbb{Z}^n$, and we put a directed edge from u to v for $u, v \in \mathbb{Z}^n$ labeled by the pair (x, y) for $x, y \in K$ if $u + x = y + Av$. The *nucleus* of the graph Γ is the subgraph (subautomaton) $\mathcal{N}$ spanned by all cycles of Γ and all vertices that can be reached following directed paths from the cycles. Since the matrix A is expanding the nucleus $\mathcal{N}$ is a finite graph and it can be algorithmically computed. Indeed, if $u + x = y + Av$ then

$$\|v\| < \|u\| \quad \text{whenever} \quad \|u\| > (1 - \|A^{-1}\|)^{-1} \max_{x,y \in K} \|A^{-1}(x - y)\|,$$

and the nucleus $\mathcal{N}$ is contained in the ball centered at the origin of radius given by the right-hand side above. Remove every edge in $\mathcal{N}$ whose label is not in $C \times D$, and replace every label (a, b) by a. We get some finite graph $\mathcal{N}_D$ whose edges are labeled by elements of the set C.

Let $C^{-\omega}$ be the space of all left-infinite sequences $\ldots x_2 x_1$, $x_i \in C$, with the product topology of discrete sets. Let μ be the uniform Bernoulli measure on $C^{-\omega}$, i.e. the product measure with $\mu(x) = 1/|C|$ for every $x \in C$. For every vertex v of the graph $\mathcal{N}_D$ denote by F_v the set of all left-infinite sequences which can be read along left-infinite paths in $\mathcal{N}_D$ that end in v. The sets F_v are closed in $C^{-\omega}$, thus compact and measurable.

Theorem 1. *The Lebesgue measure of the self-affine set T is equal*

$$\lambda(T) = \sum_{v \in \mathcal{N}_D} \mu(F_v).$$

Proof. Consider the map $\Phi : K^{-\omega} \times \mathbb{Z}^n \to \mathbb{R}^n$ given by the rule

$$\Phi(\ldots x_2 x_1, v) = v + A^{-1}x_1 + A^{-2}x_2 + \ldots,$$

where $x_i \in K$ and $v \in \mathbb{Z}^n$. Since $\mathbb{Z}^n = K + A(\mathbb{Z}^n)$, the map Φ is onto (see [8] or [9, Section 6.2]). Two elements $\xi = (\ldots x_2 x_1, v)$ and $\zeta = (\ldots y_2 y_1, u)$ for $x_i, y_i \in K$ and

$v, u \in \mathbb{Z}^n$ represent the same point $\Phi(\xi) = \Phi(\zeta)$ in $\mathbb{R}^n$ if and only if there is a finite subset $B \subset \mathbb{Z}^n$ and a sequence $\{v_m\}_{m\geq 1} \in B$ such that there exists the path

$$v_m \xrightarrow{(x_m,y_m)} v_{m-1} \xrightarrow{(x_{m-1},y_{m-1})} \ldots \xrightarrow{(x_2,y_2)} v_1 \xrightarrow{(x_1,y_1)} u - v \tag{1}$$

in the graph Γ for every $m \geq 1$. Indeed, this path implies that

$$v_m + x_m + Ax_{m-1} + \ldots + A^{m-1}x_1 + A^m v = y_m + Ay_{m-1} + \ldots + A^{m-1}y_1 + A^m u. \tag{2}$$

Applying A^{-m} and using the facts that A^{-1} is contracting and the sequence $\{v_m\}_{m\geq 1}$ attains a finite number of values, we get the equality $\Phi(\xi) = \Phi(\zeta)$. For the converse, we choose v_m such that (2) holds, and using the equality $\Phi(\xi) = \Phi(\zeta)$ we get that $\{v_m\}_{m\geq 1}$ attains a finite number of values. Notice that since the set B is assumed to be finite, every element v_m lies either on a cycle or there is a directed path from a cycle to v_m. In particular, all elements v_m should belong to the nucleus $\mathcal{N}$, and we have that the elements ξ and ζ represent the same point in $\mathbb{R}^n$ if and only if there exists a left-infinite path in $\mathcal{N}$ labeled by $(\ldots x_2x_1, \ldots y_2y_1)$ and ending in $u - v$.

Take the restriction $\Phi_C : C^{-\omega} \times \mathbb{Z}^n \to \mathbb{R}^n$ of the map Φ on the subset $C^{-\omega} \times \mathbb{Z}^n$. Since $\mathbb{Z}^n = C + A(\mathbb{Z}^n)$, the map Φ_C is also onto, and this gives an encoding of points in $\mathbb{R}^n$ by elements of $C^{-\omega} \times \mathbb{Z}^n$. Consider the uniform Bernoulli measure μ on the space $C^{-\omega}$ and the counting measure on the group $\mathbb{Z}^n$, and put the product measure on the space $C^{-\omega} \times \mathbb{Z}^n$. Since the set C is a coset transversal, the push-forward of this measure under Φ_C is the Lebesgue measure on $\mathbb{R}^n$ (see [1, Proposition 25]). Hence to find the Lebesgue measure of the self-affine set T it is sufficient to find the measure of its preimage in $C^{-\omega} \times \mathbb{Z}^n$. However, T is equal to $\Phi(D^{-\omega} \times 0)$, and hence the sequence $(\ldots x_2x_1, v)$ for $x_i \in C$ and $v \in \mathbb{Z}^n$ represents a point in T if and only if there exists a left-infinite path in the nucleus $\mathcal{N}$, which ends in $-v$ and is labeled by $(\ldots x_2x_1, \ldots y_2y_1)$ for some $y_i \in D$. Hence

$$\Phi_C^{-1}(\Phi(D^{-\omega} \times 0)) = \bigcup_{v \in \mathcal{N}_D} F_v \times \{-v\}, \tag{3}$$

and the statement follows. $\square$

The Bernoulli measure of the sets F_v for any finite graph $\Gamma = (V, E)$ can be effectively computed (see [1, Section 2]). First, we can assume that the graph is left-resolving, i.e. for every vertex $v \in V$ the incoming edges to v have different labels. Indeed, for any finite graph $\Gamma = (V, E)$ there exists a left-resolving graph $\Gamma' = (V', E')$ with the property that for every $v \in V$ there exists $v' \in V'$ such that $F_v = F_{v'}$, and this graph can be easily constructed (here every vertex v' corresponds to some subset of V, see [5, Section 2.3]). For a left-resolving graph the vector $(\mu(F_v))_{v\in V}$ (if it is nonzero) is the left eigenvector of the adjacency matrix of the graph for the eigenvalue $|C| = |\det A|$. This eigenvector is uniquely defined if we know its entries $\mu(F_v)$ for vertices v in the strongly connected components without incoming edges. For every such a component Γ', we have $F_v = C^{-\omega}$ and $\mu(F_v) = 1$ for every vertex v in Γ' if inside this component every vertex has incoming edges labeled by every element of the set C, and $\mu(F_v) = 0$ otherwise. In particular, the entries $\mu(F_v)$ are rational numbers, and we recover the following result of [4].

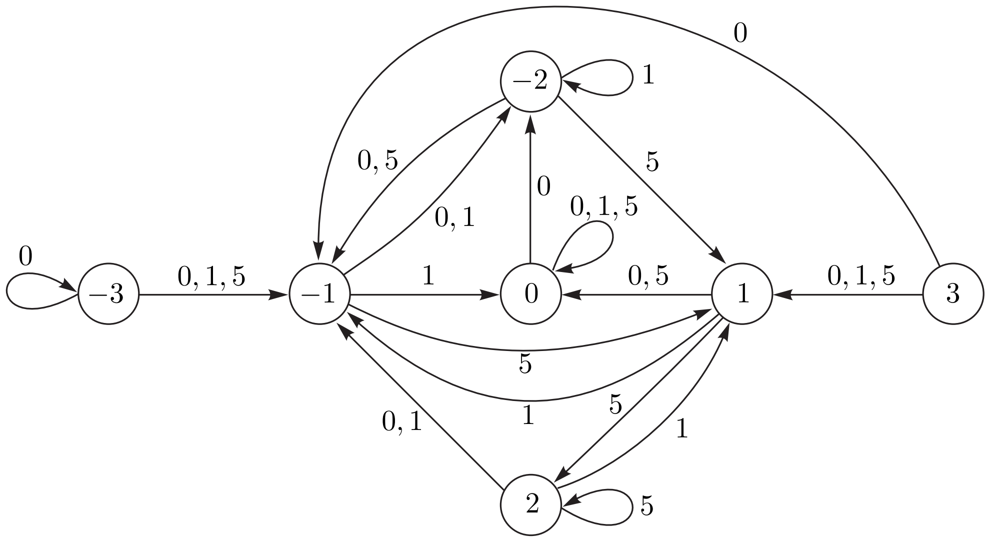

Figure 1: The graph $\mathcal{N}_D$ for $A = (3)$ and $D = \{0, 1, 5, 6\}$

Corollary 2. *Every self-affine set has rational Lebesgue measure.*

It is also easy to check when the measure of T is non-zero without calculating its precise value but just looking at the left-resolving graph (not the graph $\mathcal{N}_D$) constructed above. The measure $\lambda(T)$ will be positive if and only if there exists a strongly connected component such that inside this component every vertex has incoming edges labeled by every letter of the alphabet.

Example 1. Let $A = (3)$ and $D = \{0, 1, 5, 6\}$. The self-affine set T is $[0, \frac{4}{3}] \cup [\frac{5}{3}, 3]$, and $\lambda(T) = 8/3$. Choose $K = D$ and the coset transversal $C = \{0, 1, 5\}$. The associated automaton $\mathcal{N}_D$ is shown in Figure 1. Here $\mu(F_0) = 1$, $\mu(F_1) = 1/3$, $\mu(F_2) = 1/8$, $\mu(F_{-1}) = 7/12$, $\mu(F_{-2}) = 5/8$, and $\mu(F_{-3}) = \mu(F_3) = 0$.

The above method can be used to find $\lambda(T \cap (T + u))$ for $u \in \mathbb{Z}^n$. The set $T + u$ is the image of the set $D^{-\omega} \times u$, and its preimage under Φ_C can be described as in (3). In particular

$$\lambda(T \cap (T + u)) = \sum_{\substack{v_1, v_2 \in \mathcal{N}_D \\ u = v_2 - v_1}} \mu(F_{v_1} \cap F_{v_2}).$$

Similarly, one can find the measure of the intersection of self-affine sets $T_1 = T(A, D_1)$ and $T_2 = T(A, D_2)$ for different sets $D_1, D_2 \subset \mathbb{Z}^n$. We take a set E which contains D_1, D_2, and some coset transversal C, and as above we construct the nucleus $\mathcal{N}$ and its subgraphs $\mathcal{N}_{D_1}$ and $\mathcal{N}_{D_2}$. Then

$$\lambda(T_1 \cap T_2) = \sum_{v \in \mathcal{N}} \mu(F_v^{(1)} \cap F_v^{(2)}),$$

where $F_v^{(i)}$ is calculated in the graph $\mathcal{N}_{D_i}$. Hence these two problems are reduced to the question of how to find the measure of the intersection $F_{v_1}^{(1)} \cap F_{v_2}^{(2)}$, where each set $F_{v_i}^{(i)}$ is defined in some finite graph $\Gamma^{(i)} = (V^{(i)}, E^{(i)})$ with its vertex v_i. One can construct a new finite graph Γ (sometimes called the labeled product of graphs $\Gamma^{(i)}$) with the set of vertices $V^{(1)} \times V^{(2)}$, where we put an edge $(u_1, u_2) \xrightarrow{x} (w_1, w_2)$ for every edges $u_1 \xrightarrow{x} w_1$ in $\Gamma^{(1)}$ and $u_2 \xrightarrow{x} w_2$ in $\Gamma^{(2)}$. Then $F_{(v_1,v_2)} = F_{v_1}^{(1)} \cap F_{v_2}^{(2)}$ (see [5, Section 3.2]).

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